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Dense Markov Spaces and Unbounded Bernstein Inequalities*

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An infinite Markov system $\{f_0, f_1, ...\}$ of C^2 functions on $[a, b]$ has dense span in $C[a, b]$ if and only if there is an unbounded Bernstein inequality on every subinterval of $[a, b]$. That is if and only if, for each $[\alpha, \beta] \subset [a, b]$, $\alpha \neq \beta$, and $\gamma > 0$, we can find $g \in \text{span}\{f_0, f_1, ...\}$ with $\|g'\|_{[\alpha, \beta]} > \gamma \|g\|_{[a,b]}$. This is proved under the assumption $(f_1/f_0)'$ does not vanish on (a, b). Extension to higher derivatives are also considered. An interesting consequence of this is that functions in the closure of the span of a non-dense C^2 Markov system are always C^∞ on some subinterval.

The principal result of this paper will be a characterization of denseness of the span of a Markov system by whether or not it possesses an unbounded Bernstein Inequality. In order to make sense of this result we require the following definitions.

DEFINITION 1 (Chebyshev System). Let $f_0, ..., f_n$ be elements of $C[a, b]$ the real valued continuous functions on $[a, b]$. Suppose that $\text{span}\{f_0, ..., f_n\}$ over $\mathbb{R}$ is an $n+1$ dimensional subspace of $C[a, b]$. Then $\{f_0, ..., f_n\}$ is called a Chebyshev system of dimension $n+1$ on $[a, b]$ if any element of $\text{span}\{f_0, ..., f_n\}$ that has $n+1$ distinct zeros in $[a, b]$ is identically zero. If $\{f_0, ..., f_n\}$ is a Chebyshev system, then $\text{span}\{f_0, ..., f_n\}$ is called a Chebyshev space.

DEFINITION 2 (Markov System). We say that $\{f_0, ..., f_n\}$ is a Markov system on $[a, b]$ if each $f_i \in C[a, b]$ and $\{f_0, ..., f_m\}$ is a Chebyshev system

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on $[a, b]$ for every $m \geqslant 0$. (We allow n to tend $+\infty$ in which case we call the system an infinite Markov system.)

DEFINITION 3 (Unbounded Bernstein Inequality). Let $\mathscr{A}$ be a subset of $C^1[a, b]$. We say that $\mathscr{A}$ has an everywhere unbounded Bernstein inequality if for every $[\alpha, \beta] < [a,b]$, $\alpha \neq \beta$,

$$\sup \left\{ \frac{\|p'\|_{[\alpha,\beta]}}{\|p\|_{[a,b]}} : p \in \mathscr{A}, p \neq 0 \right\} = \infty.$$

If for some $[\alpha, \beta]$ the above sup is finite, the Bernstein inequality is said to be bounded in $[\alpha, \beta]$.

Note that the collection of all polynomials of the form

$$\{x^2 p(x) : p \text{ is a polynomial}\}$$

has an everywhere unbounded Bernstein inequality on $[-1, 1]$ despite the fact that every element has derivative vanishing at zero.

We now state the main result.

THEOREM 1. *Suppose* $\mathscr{M} := \{f_0, f_1, f_2, ...\}$ *is an infinite Markov system on* $[a, b]$ *with each* $f_i \in C^2[a, b]$, *and suppose that* $(f_1/f_0)'$ *does not vanish on* (a, b). *Then* span $\mathscr{M}$ *is dense in* $C[a, b]$ *if and only if* span $\mathscr{M}$ *has an everywhere unbounded Bernstein inequality.*

The additional assumption that $(f_1/f_0)'$ does not vanish on (a, b) is quite weak. It holds, for example, for any ECT system. Note that f_1/f_0 is strictly monotone if $\mathscr{M}$ is a Markov system.

The proof requires examining the Chebyshev polynomials associated with a Chebyshev system. These we now discuss.

Suppose

$$H_n := \text{span}\{f_0, ..., f_n\}$$

is a Chebyshev space on $[a, b]$. We can define the Chebyshev polynomial

$$T_n(x) := T_n\{f_0, ..., f_n; [a, b]\}(x)$$

associated with H_n as the unique "generalized" polynomial in $\operatorname{span}\{f_0, ..., f_n\}$ that alternates between ± 1 exactly $n+1$ times with $T_n(b) > 0$, and has exactly n zeros on $[a, b]$. With $f_i := x^i$, this generates the usual Chebyshev polynomials. These equioscillating polynomials encode much of the information of how the space H_n behaves with respect to the supremum norm. See [2, 3, 4, and 6].

Suppose

$$\mathcal{M} = \{f_0, f_1, ...\}$$

is a fixed infinite Markov system on $[a, b]$. For each n

$$H_n := \{f_0, f_1, ..., f_n\}$$

is then a Chebyshev system. So there is a sequence $\{T_n\}$ of associated Chebyshev polynomials where, for each n, T_n is associated with H_n. These we call the associated Chebyshev polynomials for the infinite Markov system $\mathcal{M}$.

Note that

$$\{T_0, T_1, ...\}$$

is a Markov system again with the same span as $\mathcal{M}$.

In [2] we showed that the span of a C^1 Markov system $\mathcal{M}$ is dense in $C[a, b]$ in the uniform norm (i.e., the uniform closure of span $\mathcal{M}$ on $[a, b]$ equals $C[a, b]$) if and only if the zeros of the associated Chebyshev polynomials are dense. To state this result, which we will need, we require the following notation.

Suppose T_n has zeros $a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b$, and let $x_0 := a$ and $x_{n+1} := b$. Then the mesh of T_n is defined by

$$M_n := M_n(T_n : [a, b])$$

$$:= \max_{1 \leqslant i \leqslant n+1} |x_i - x_{i-1}|.$$

For a sequence of Chebyshev polynomials T_n from a fixed Markov system on $[a, b]$, we have

$$M_n \to 0 \qquad \text{iff} \quad \underline{\lim}\, M_n = 0$$

as follows from the interlacing of the zeros of T_n and T_m.

Our main result requires the following theorem from [2].

THEOREM 2. *Suppose* $\mathcal{M} := \{1, f_1, f_2, ...\}$ *is an infinite Markov system on* $[a, b]$ *with each* $f_i \in C^1[a, b]$. *Then* span $\mathcal{M}$ *is dense in* $C[a, b]$ *in the uniform if and only if*

$$M_n \to 0,$$

(*where* M_n *is the mesh of the associated Chebyshev polynomials*).

The next result we need shows that in most instances the Chebyshev polynomial is close to extremal for Bernstein-type inequalities.

THEOREM 3. *Let* $H_n := \{1, f_1, ..., f_n\}$ *be a Chebyshev system of* C^1 *functions on* $[a, b]$. *Let* T_n *be the associated Chebyshev polynomial. Then*

$$\frac{|p_n'(x_0)|}{\|p_n\|_{[a,b]}} \leqslant \frac{2}{1 - |T_n(x_0)|} |T_n'(x_0)|$$

for every $0 \neq p_n \in \text{span}\{1, f_1, ..., f_n\}$ *and every* $x_0 \in [a, b]$ *with* $|T_n(x_0)| \neq 1$.

Proof. Let $a = y_0 < y_1 < \cdots < y_n = b$ denote the extreme points of T_n, so

$$T_n(y_i) = (-1)^{n-i}, \qquad i = 0, 1, ..., n.$$

Let $y_k \leqslant x_0 \leqslant y_{k+1}$ and $0 \neq p_n \in H_n$. If $p_n'(x_0) = 0$, then there is nothing to prove. So assume that $p_n'(x_0) \neq 0$. Then we may normalize p_n so that

$$\|p_n\|_{[a,b]} = 1$$

and

$$\text{sign}(p_n'(x_0)) = \text{sign}(T_n(y_{k+1}) - T_n(y_k)).$$

Let $\delta := |T_n(x_0)|$. Let $\varepsilon \in (0, 1)$ be fixed. Then there exists a constant η with $|\eta| \leqslant \delta + (1 - \delta)/2$ so that

$$\eta + \frac{(1-\delta)}{2}(1 - \varepsilon) p_n(x_0) = T_n(x_0).$$

Now let

$$q_n(x) := \eta + \frac{(1-\delta)}{2}(1 - \varepsilon) p_n(x).$$

Then

$$\|q_n\|_{[a,b]} < 1,$$

$$q_n(x_0) = T_n(x_0),$$

and

$$\text{sign}(q_n'(x_0)) = \text{sign}(T_n(y_{k+1}) - T_n(y_k)).$$

If the desired inequality does not hold for p_n then for a sufficiently small $\varepsilon > 0$

$$|q_n'(x_0)| > |T_n'(x_0)|,$$

so

$$h_n(x) := q_n(x) - T_n(x)$$

will have at least three zeros in (y_k, y_{k+1}). But h_n has at least one zero in each of (y_i, y_{i+1}). Hence $h_n \in H_n$ has at least $n+2$ zeros in $[a, b]$, which is a contradiction. ∎

We need the following technical result concerning Chebyshev polynomials.

LEMMA 1. *Suppose* $\mathcal{M} := \{1, f_1, f_2, ...\}$ *is an infinite Markov system of* C^2 *functions on* $[a, b]$ *and* f_1' *does not vanish on* (a, b). *Suppose that the sequence of the associated Chebyshev polynomials* $\{T_n\}$ *has a subsequence* $\{T_{n_i}\}$ *with no zeros on some subinterval of* $[a, b]$. *Then there exists another subinterval* $[c, d]$ *and another infinite subsequence* $\{T_{n_i}\}$ *so that for some* $\delta > 0$, $\gamma > 0$, *and for each* n_i,

$$\|T_{n_i}\|_{[c,d]} < 1 - \delta$$

and

$$\|T_{n_i}'\|_{[c,d]} < \gamma.$$

Proof. For both inequalities we first choose a subinterval $[c_1, d_1] \subset [a, b]$ and a subsequence $\{n_{i,1}\}$ of $\{n_i\}$ so that all oscillations of each $T_{n_{i,1}}$ take place away from $[c_1, d_1]$. We now choose a subsequence $\{n_{i,2}\}$ of $\{n_{i,1}\}$ so that either each $T_{n_{i,2}}$ is increasing or each $T_{n_{i,2}}$ is decreasing on $[c_1, d_1]$. We treat the first case, the second one is analogous. Let $[c_2, d_2]$ be the middle third of $[c_1, d_1]$. If the first inequality fails to hold with $[c_2, d_2]$ and $\{n_{i,2}\}$ then there is a subsequence $\{n_{i,3}\}$ of $\{n_{i,2}\}$ so that

$\|T_{n_{i,3}}\|_{[c_2,d_2]} \to 1$ as $n_{i,3} \to \infty$. Hence, there is a subsequence $\{n_{i,4}\}$ of $\{n_{i,3}\}$ so that either

$$\max_{c_2 \leqslant x \leqslant d_2} T_{n_{i,4}}(x) \to 1 \qquad \text{or} \qquad \min_{c_2 \leqslant x \leqslant d_2} T_{n_{i,4}}(x) \to -1.$$

Once again we treat the fist case, the second one is analogous. Since each $T_{n_{i,3}}$ is increasing on $[c_1, d_1]$,

$$\lim_{n_{i,4} \to \infty} \|1 - T_{n_{i,4}}\|_{[d_2, d_1]} = 0.$$

Now take $g := a_0 + a_1 f_1 + a_2 f_2$ so that g has two distinct zeros α_1 and α_2 in $[d_2, d_1]$, $\|g\|_{[\alpha_1, \alpha_2]} < 1$, and g is positive on (α_1, α_2). Let $\beta := \max_{\alpha_1 \leqslant x \leqslant \alpha_2} g(x)$ and $\tilde{g} := g + 1 - \beta$. One can now deduce that $T_{n_{i,4}} - \tilde{g}$ has at least $n+1$ distinct zeros in $[a, b]$ if $n_{i,4}$ is large enough, which is a contradiction.

For the second inequality, by [8], $\{f'_1, f'_2, ...\}$ is a weak Markov system on $[a, b]$, and so is

$$\{(T'_2/T'_1)', (T'_3/T'_1)', ...\}$$

on every closed subinterval of (a, b). (In the definitions of weak Markov systems and weak Chebyshev systems we only count zeros where the sign changes.) The assumption that f'_1 does not vanish on (a, b) implies that T'_1 does not vanish on (a, b).

From this we deduce that each $(T'_{n_{i,2}}/T'_1)'$ has at most one sign change in $[c_2, d_2]$. Choose a subinterval $[c_3, d_3] \subset [c_2, d_2]$ and a subsequence $\{n_{i,5}\}$ of $\{n_{i,2}\}$ so that none of $(T'_{n_{i,5}}/T'_1)'$ changes sign in $[c_3, d_3]$. Choose a subsequence $\{n_{i,6}\}$ of $\{n_{i,5}\}$ so that either each $T'_{n_{i,6}}/T'_1$ is increasing or each $T'_{n_{i,6}}/T'_1$ is decreasing on $[c_3, d_3]$. We only study the first case; the second one is similar. Let $[c_4, d_4]$ be the middle third of $[c_3, d_3]$. If the second inequality fails to hold with $[c_4, d_4]$ and $\{n_{i,6}\}$ then there is a subsequence $\{n_{i,7}\}$ of $\{n_{i,6}\}$ so that either

$$\max_{c_4 \leqslant x \leqslant d_4} T'_{n_{i,7}}(x)/T'_1(x) \to \infty$$

or

$$\min_{c_4 \leqslant x \leqslant d_4} T'_{n_{i,7}}(x)/T'_1(x) \to -\infty$$

as $n_{i,7} \to \infty$. Again we treat only the first case, the second one is analogous. Then for every $K > 0$ there is $N \in \mathbb{N}$ so that for every $n_{i,7} \geqslant N$ we have

$$T'_{n_{i,7}}(x) > K, \qquad x \in [d_4, d_3],$$

hence

$$K(d_3 - d_4) \leqslant \int_{d_4}^{d_3} T'_{n_i,\gamma}(x)\,dx = T_{n_i,\gamma}(d_3) - T_{n_i,\gamma}(d_4) \leqslant 2,$$

which is a contradiction. ∎

LEMMA 2. *Suppose $\mathcal{M} := \{f_0, f_1, ...\}$ is a $C^1[a, b]$ infinite Markov system, and suppose $g \in C^1[a, b]$ and g is strictly positive on $[a, b]$. Then $\mathcal{N} = \{gf_0, gf_1, ...\}$ is also a $C^1[a, b]$ infinite Markov system. Furthermore* span $\mathcal{M}$ *has a bounded Bernstein inequality on $[\alpha, \beta] \subset [a, b]$ if and only if* span $\mathcal{N}$ *also has bounded Bernstein inequality on $[\alpha, \beta]$.*

Proof. Consider differentiating gf with $f \in \operatorname{span} \mathcal{M}$ by the product rule. If span $\mathcal{M}$ has a bounded Bernstein inequality on $[\alpha, \beta]$ then

$$\begin{aligned} \|(gf)'\|_{[\alpha,\beta]} &\leqslant \|g'f\|_{[\alpha,\beta]} + \|gf'\|_{[\alpha,\beta]} \\ &\leqslant c_1 \|gf\|_{[\alpha,\beta]} + c_2 \|gf\|_{[a,b]}, \end{aligned}$$

where the first constant arises since

$$g'(x)/g(x)$$

is uniformly bounded on $[a, b]$ and the second constant comes from the bounded Bernstein inequality for f. ∎

Proof of Theorem 1. The only if part of this theorem is obvious. A good uniform approximation to a function with uniformly large derivative on a subinterval $[\alpha, \beta] \subset [a, b]$ must have large derivative at some points in $[\alpha, \beta]$.

In the other direction we first note that by Lemma 2 we may assume $f_0 = 1$. We use Theorem 2 and Lemma 1 in the following way. If span $\mathcal{M}$ is not dense then there exists a subinterval $[\alpha, \beta] \subset [a, b]$ by Theorem 2, where a subsequence of the associated Chebyshev polynomials have no zeros. By Lemma 1 from this subsequence we can pick another subsequence T_{n_i} and a subinterval $[c, d] \subset [\alpha, \beta]$ with

$$\|T_{n_i}\|_{[c,d]} < 1 - \delta$$

and

$$\|T'_{n_i}\|_{[c,d]} < \gamma$$

for some positive constants δ and γ. The result now follows from Theorem 3. ∎

COROLLARY 1. *Suppose $\mathcal{M}=\{f_0, f_1, ...\}$ is an infinite Markov system of C^2 functions on $[a, b]$ so that* span $\mathcal{M}$ *fails to be dense in $C[a, b]$ in the uniform norm. Then there exists a subinterval $[\alpha, \beta]$ of $[a, b]$ so that if g is in the uniform closure of* span $\mathcal{M}$ *then g is differentiable on $[\alpha, \beta]$.*

Proof. By Theorem 1, there exists an interval $[\alpha, \beta]$ where $\|h'\|_{[\alpha,\beta]}/\|h\|_{[\alpha,\beta]}$ is uniformly bounded for every $h \in \operatorname{span} \mathcal{M}$. Suppose $h_n \to g$, $h_n \in \operatorname{span} \mathcal{M}$. Then we can choose n_i so that

$$\|g-h_{n_i}\|_{[a,b]} \leqslant \frac{1}{2^i}, \qquad i=0, 1, 2, ...$$

and hence

$$g=h_{n_0}+\sum_{i=1}^{\infty} (h_{n_i}-h_{n_{i-1}}).$$

Since

$$\|(h_{n_i}-h_{n_{i-1}})'\|_{[\alpha,\beta]} \leqslant \frac{c}{2^i}$$

for some constant c independent of i, if follows that g is differentiable on $[\alpha, \beta]$. ∎

Suppose $\mathcal{M}=\{f_0, f_1, ...\}$ is an extended complete Markov system of C^∞ functions on $[a, b]$ (the extra requirement being that the multiplicity of the zeros matters in the definition: so if $f:=\sum_{i=0}^{n} a_i f_i$ has $n+1$ zeros by counting multiplicities then $f=0$ identically). In this case the differential operator D defined by

$$D(f):=\left(\frac{f}{f_0}\right)'$$

maps $\mathcal{M}$ to $\mathcal{M}_D$ where

$$\mathcal{M}_D=\left\{\left(\frac{f_1}{f_0}\right)', \left(\frac{f_2}{f_0}\right)', ...\right\}$$

and $\mathcal{M}_D$ is once again an extended complete Markov system of C^∞ functions (see Nürnberger [5]). We define the differential operators $D^{(n)}(f)$ for n times differentiable functions f by

$$F_{i,0}:=f_i, \qquad F_{i,n}:=\left(\frac{F_{i+1,n-1}}{F_{0,n-1}}\right)', \quad i=0, 1, ..., \quad n=1, 2, ...,$$

$$D^{(0)}(f):=f, \qquad D^{(n)}(f):=\left(\frac{D^{(n-1)}(f)}{F_{0,n-1}}\right)', \quad n=1, 2, ...$$

Note that if span $\mathcal{M}_D$ is dense in $C[a, b]$ in the uniform norm then so is span $\mathcal{M}$. The next theorem can be obtained from Theorem 1 by induction on n.

THEOREM 4. *Suppose $\mathcal{M} = \{f_0, f_1, ...\}$ is an extended complete Markov system of C^∞ functions on $[a, b]$. Let n be a fixed positive integer. Suppose* span $\mathcal{M}$ *fails to be dense in $C[c, d]$ for every subinterval $[c, d] \subset [a, b]$, $c \neq d$. Then there exists an interval $[\alpha_n, \beta_n] \subset [a, b]$, $\alpha_n \neq \beta_n$, so that*

$$\sup\left\{\frac{\|D^{(n)}(f)\|_{[\alpha_n, \beta_n]}}{\|f\|_{[a,b]}} : 0 \neq f \in \operatorname{span} \mathcal{M}\right\} < \infty.$$

COROLLARY 2. *Suppose $\mathcal{M}$ is an extended complete Markov system of C^∞ functions on $[a, b]$ so that* span $\mathcal{M}$ *fails to be dense in $C[a, b]$ in the uniform norm. Then for each n there exists an interval $[\alpha_n, \beta_n] \subset [a, b]$ of positive length where all elements of the uniform closure of* span $\mathcal{M}$ *are n times continuously differentiable.*

Proof. Use Theorem 4 as in Corollary 1. We omit the technical details. ∎

Suppose that $\mathcal{M}$, as in Corollary 2, has the property that span $\mathcal{M}$ fails to be dense in the uniform norm on any proper subinterval of $[a, b]$, as in the case of Müntz systems

$$\mathcal{M} := \{x^{\lambda_0}, x^{\lambda_1}, ...\}, \qquad 0 \leqslant \lambda_0 < \lambda_1 < \cdots, \qquad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty, \qquad 0 \leqslant a < b.$$

Then the uniform closure of span $\mathcal{M}$ on $[a, b]$ contains only functions that are C^∞ on a dense subset of $[a, b]$. In this non-dense Müntz case the closure actually contains only analytic functions on (a, b) (Achiezer [1], Schwartz [7]).

We record one final corollary.

COROLLARY 3. *Suppose $\{\alpha_k\} \subset \mathbb{R} \backslash [-1, 1]$ is a sequence of distinct numbers. Then*

$$\operatorname{span}\left\{1, \frac{1}{x - \alpha_1}, \frac{1}{x - \alpha_2}, ...\right\}$$

is dense in $C[-1, 1]$ if and only if

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty.$$

Proof. The inequality

$$|p'(x)| \leqslant \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^{n} \frac{\sqrt{\alpha_k^2-1}}{|\alpha_k - x|} \|p\|_{[-1,1]}$$

holds for any

$$p \in \operatorname{span}\left\{1, \frac{1}{x-\alpha_1}, ..., \frac{1}{x-\alpha_n}\right\}.$$

See [3]. This together with Theorem 1 gives the "only if" part of the corollary.

In [3] the Chebyshev "polynomials" T_n (of the first kind) and U_n (of the second kind) for the Chebyshev space

$$\operatorname{span}\left\{1, \frac{1}{x-\alpha_1}, ..., \frac{1}{x-\alpha_n}\right\}$$

are introduced. Properties of

$$\tilde{T}_n(t) := T_n(\cos t)$$

and

$$\tilde{U}_n(t) := U_n(\cos t) \sin t$$

established in [3] include

$$\|\tilde{T}_n\|_{\mathbb{R}} = 1 \qquad \text{and} \qquad \|\tilde{U}_n\|_{\mathbb{R}} = 1, \tag{1}$$

$$\tilde{T}_n(t)^2 + \tilde{U}_n(t)^2 = 1, \qquad t \in \mathbb{R}, \tag{2}$$

$$\tilde{T}'_n(t)^2 + \tilde{U}'_n(t)^2 = \tilde{B}_n(t)^2, \qquad t \in \mathbb{R}, \tag{3}$$

$$\tilde{T}'_n(t) = -\tilde{B}_n(t)\, \tilde{U}_n(t), \qquad t \in \mathbb{R}, \tag{4}$$

$$\tilde{U}'_n(t) = \tilde{B}_n(t)\, \tilde{T}_n(t), \qquad t \in \mathbb{R}, \tag{5}$$

where

$$\tilde{B}_n(t) = \sum_{k=1}^{n} \frac{\sqrt{\alpha_k^2-1}}{|\alpha_k - \cos t|}, \qquad t \in \mathbb{R}.$$

Suppose

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty.$$

Then

$$\lim_{n\to\infty} \min_{t\in[\alpha,\beta]} \tilde{B}_n(t) = \infty, \qquad 0<\alpha<\beta<\pi. \tag{6}$$

Assume that there is a subinterval $[a, b]$ of $(-1, 1)$ so that

$$\sup_{n\in\mathbb{N}} \|T'_n\|_{[a,b]} < \infty.$$

Let $\alpha := \arccos b$ and $\beta := \arccos a$. Then by properties (4) and (6)

$$\lim_{n\to\infty} \|\tilde{U}_n\|_{[\alpha,\beta]} = 0$$

hence by property (2)

$$\lim_{n\to\infty} \|\tilde{T}_n^2 - 1\|_{[\alpha,\beta]} = 0.$$

Thus by properties (5) and (6)

$$\lim_{n\to\infty} \min_{t\in[\alpha,\beta]} |\tilde{U}'_n(t)| = \infty$$

that is

$$\lim_{n\to\infty} |\tilde{U}_n(\beta) - \tilde{U}_n(\alpha)| = \infty$$

which contradicts property (1). Hence

$$\sup_{n\in\mathbb{N}} \frac{\|T'_n\|_{[a,b]}}{\|T_n\|_{[-1,1]}} = \sup_{n\in\mathbb{N}} \|T'_n\|_{[a,b]} = \infty.$$

for every subinterval $[a, b]$ of $(-1, 1)$ which together with Theorem 1 finishes the "if" part of the proof. ∎

Corollary 3 is to be found in Achieser [1, p. 255] proven by entirely different methods.

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